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Gröbner bases in difference-differential modules and difference-differential dimension polynomials

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Abstract In this paper we extend the theory of Gröbner bases to difference-differential modules and present a new algorithmic approach for computing the Hilbert function of a finitely generated difference-differential module equipped with the natural filtration. We present and verify algorithms for constructing these Gröbner bases counterparts. To this aim we introduce the concept of "generalized term order" on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules. Using Gröbner bases on difference-differential modules we present a direct and algorithmic approach to computing the difference-differential dimension polynomials of a difference-differential module and of a system of linear partial difference-differential equations.

Keywords: Gröbner basis, generalized term order, difference-differential module, difference-differential dimension polynomial

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1 Introduction

The efficiency of the classical Gröbner basis method for the solution of problems by algorithmic way in polynomial ideal theory is well-known. The results of Buchberger[1] on Gröbner bases in polynomial rings have been generalized by many researchers to non-commutative case, especially to modules over various rings of differential operators. Galligo[2] first gave the Gröbner basis algorithm for the Weyl algebra A_n. Mora[3] generalized the concept of Gröbner basis to non-commutative free algebras. Noumi[4] and Takayama[5] formulated the Gröbner bases in R_n, the ring of differential operators with rational function coefficients. Oaku and Shimoyama[6] treated D_0, the ring of differential operators with power series coefficients. Insa and Pauer[7] presented a basic theory of Gröbner bases for differential operators with coefficients in a commutative noetherian ring. It has been proved that the notion of Gröbner basis is a powerful tool to solve various problems of linear partial differential equations.

On the other hand, for some problems of linear difference-differential equations such as the dimension of the space of solutions and the computation of difference-differential dimension polynomials, the notion of Gröbner basis for the ring of difference-differential operators is essential. Gröbner bases in rings of differential operators are defined with respect to a term order

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on $\mathbb{N}^n \times \mathbb{N}^n$ or $\mathbb{N}^n$. This approach cannot be used for the ring of difference-differential operators, because for it we need to treat orders on $\mathbb{N}^m \times \mathbb{Z}^n$. Pauer and Unterkircher[8] considered Gröbner bases in Laurent polynomial rings, but it is limited in commutative case. Levin[9] introduced a characteristic set for free modules over rings of difference-differential operators. It is an analog of "Gröbner basis" connected with a specific "ordering" on $\mathbb{N}^m \times \mathbb{Z}^n$. But the ordering is not a term-ordering while the theory of Gröbner basis works for any term-ordering.

The concept of the differential dimension polynomial was introduced by Kolchin[10] as a dimensional description of some differential field extension. Johnson[11] proved that the differential dimension polynomial of a differential field extension coincides with the Hilbert polynomial of some filtered differential module. This result allowed to compute differential dimension polynomials using the Gröbner basis technique. Since then various problems of differential algebra involving differential dimension polynomials were studied (see [12, 13]). The concepts of the difference dimension polynomial and the difference-differential dimension polynomial were introduced first in [14, 15] respectively. They play the same role in difference algebra (resp. difference-differential algebra) as Hilbert polynomials in commutative algebra or differential dimension polynomials in differential algebra. The notion of difference-differential dimension polynomial can be used for the study of dimension theory of difference-differential field extension and of systems of algebraic difference-differential equations.

Mikhalev and Pankratev[15] proved existence of difference-differential dimension polynomials $\phi(t)$ associated with a difference-differential module M by classical Gröbner basis methods of computation of Hilbert polynomials. The proof is based on the fact that the ring of difference-differential operators over the difference-differential field R is isomorphic to the factor ring of the ring of generalized polynomials $R[x_1, \dots, x_{m+2n}]$ (where $x_i a = a x_i + \delta_i(a)$ $(1 \leqslant i \leqslant m)$, $x_{m+j} a = \alpha_j(a) x_{m+j}$ and $x_{m+n+j} a = \alpha_j^{-1}(a) x_{m+n+j}$ $(1 \leqslant j \leqslant n)$ for any $a \in R$) by the ideal I generated by the polynomials $x_{m+j} x_{m+n+j} - 1$ $(1 \leqslant j \leqslant n)$. Wu[16] also computed dimensions of linear difference-differential systems by the above approach. However, a similar approach to difference-differential dimension polynomials in two variables is unsuccessful. Levin[9] investigated the difference-differential dimension polynomials in two variables by the characteristic set approach. The method of Levin is rather delicate but no general algorithm for computing the characteristic set.

In this paper we present a new approach, difference-differential Gröbner basis, for algorithmic computing the difference-differential dimension polynomials. It is based on the algorithm of computation of Gröbner basis for an ideal of (or a module over) the ring of difference-differential operators. In Section 2 the generalized term order and its properties are discussed and some examples are presented. In Section 3 we design carefully the reduction algorithm, the definition of the Gröbner basis and the S-polynomials, as well as the Buchberger algorithm for the computation of the Gröbner bases. In Section 4 we describe an approach to computing difference-differential dimension polynomials associated with a module over the ring of difference-differential operators via the Gröbner bases.

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_-$ and $\mathbb{Q}$ denotes the sets of all integers, all non-negative integers, all non-positive integers, and all rational numbers, respectively. By a ring we always mean an associative ring with a unit. By the module over a ring A we mean a unitary left

A-module.

Definition 1.1. *Let R be a commutative noetherian ring, $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ be set of derivations and automorphisms of the ring R, respectively, such that $\beta(x) \in R$ and $\beta(\gamma(x)) = \gamma(\beta(x))$ hold for any $\beta, \gamma \in \Delta \cup \sigma$ and $x \in R$. Then R is called a difference-differential ring with the basic set of derivations Δ and the basic set of automorphisms σ, or shortly a Δ-σ-ring. If R is a field, then it is called a Δ-σ-field.*

If R is a Δ-σ-ring, then Λ will denote the commutative semigroup of elements of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n}, \tag{1.1}$$

where $(k_1, \ldots, k_m) \in \mathbb{N}^m$ and $(l_1, \ldots, l_n) \in \mathbb{Z}^n$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative semigroup Γ generated by the set σ. The subset $\{\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1}\}$ of Λ will be denoted by σ^*.

Definition 1.2. *Let R be a Δ-σ-ring and the semigroup Λ be as above. Then an expression of the form*

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{1.2}$$

where $a_\lambda \in R$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero, is called a difference-differential operator (or shortly a Δ-σ-operator) over R. Two Δ-σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ and $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ are equal if and only if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda$.

The set of all Δ-σ-operators over a Δ-σ-ring R is a ring with the following fundamental relations

$$\begin{aligned} &\sum_{\lambda \in \Lambda} a_\lambda \lambda + \sum_{\lambda \in \Lambda} b_\lambda \lambda = \sum_{\lambda \in \Lambda} (a_\lambda + b_\lambda)\lambda, \quad a\Bigg(\sum_{\lambda \in \Lambda} a_\lambda \lambda\Bigg) = \sum_{\lambda \in \Lambda} (a a_\lambda)\lambda, \\ &\Bigg(\sum_{\lambda \in \Lambda} a_\lambda \lambda\Bigg)\mu = \sum_{\lambda \in \Lambda} a_\lambda(\lambda\mu), \quad \delta a = a\delta + \delta(a), \qquad \tau a = \tau(a)\tau, \end{aligned} \tag{1.3}$$

for all $a_\lambda, b_\lambda \in R$, $\lambda, \mu \in \Lambda$, $a \in R$, $\delta \in \Delta$, $\tau \in \sigma^*$. Note that the elements in Δ and σ^* do not commute with the elements in R, and then the "terms" $\lambda \in \Lambda$ do not commute with the coefficients $a_\lambda \in R$.

Definition 1.3. *The ring of all Δ-σ-operators over a Δ-σ-ring R is called the ring of difference-differential operators (or shortly the ring of Δ-σ-operators) over R, which will be denoted by D. A left D-module M is called a difference-differential module (or a Δ-σ-module). If M is finitely generated as a left D-module, then M is called a finitely generated Δ-σ-module.*

When $\sigma = \emptyset$, D will be the rings of differential operators $R[\delta_1, \ldots, \delta_m]$. If the coefficient ring R is the polynomial ring over a field K, then D will be Weyl algebra A_m. So Δ-σ-module is a generalization of module over rings of differential operators. But in the ring of Δ-σ-operators the "terms" are the form (1.1) and the index in $\alpha_1, \ldots, \alpha_n$ is $(l_1, \ldots, l_n) \in \mathbb{Z}^n$, the notion of "term order", as commonly used on Gröbner bases theory, is no longer valid. We generalize the concept of term order in the following section.

2 Generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$

Note that $\mathbb{Z}^n = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is a group. We consider first some decomposition of $\mathbb{Z}^n$. For instance, $\mathbb{Z}^2 = \{\mathbb{N} \times \mathbb{N}\} \cup \{\mathbb{N} \times \mathbb{Z}_-\} \cup \{\mathbb{Z}_- \times \mathbb{N}\} \cup \{\mathbb{Z}_- \times \mathbb{Z}_-\}$. Then we may define a kind of "term order" along every one of the 4 directions. And then an algorithm based on the "term order" may terminate after finite steps.

Definition 2.1. *Let $\mathbb{Z}^n$ be the union of finitely many subsets $\mathbb{Z}_j^{(n)}$: $\mathbb{Z}^n = \bigcup_{j=1}^k \mathbb{Z}_j^{(n)}$ where $\mathbb{Z}_j^{(n)}$, $j = 1, \ldots, k$, satisfy the following conditions*:

(i) $(0, \ldots, 0) \in \mathbb{Z}_j^{(n)}$, *and $\mathbb{Z}_j^{(n)}$ does not contain any pair of invertible elements $c = (c_1, \ldots, c_n) \neq 0$ and $c^{-1} = (-c_1, \ldots, -c_n)$;*

(ii) *$\mathbb{Z}_j^{(n)}$ is finitely generated sub-semigroup of $\mathbb{Z}^n$;*

(iii) *the group generated by $\mathbb{Z}_j^{(n)}$ is $\mathbb{Z}^n$,*

then $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ is called an orthant decomposition of $\mathbb{Z}^n$ and $\mathbb{Z}_j^{(n)}$ is called the j-th orthant of the decomposition.

Remark. The conditions in Def. 2.1 imply that we may define a smallest element in every $\mathbb{Z}_j^{(n)}$, and that $\mathbb{Z}_j^{(n)}$ has some similar structures as $\mathbb{N}^{(n)}$.

Example 2.2. Let $\{\mathbb{Z}_1^{(n)}, \ldots, \mathbb{Z}_{2^n}^{(n)}\}$ be all distinct Cartesian products of n sets each of which is either $\mathbb{N}$ or $\mathbb{Z}_-$. Then it is an orthant decomposition of $\mathbb{Z}^n$. The set of generators of $\mathbb{Z}_j^{(n)}$ as a semigroup is $\{(c_1, 0, \ldots, 0), (0, c_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, c_n)\}$, where c_i is either 1 or -1, $i = 1, \ldots, n$. We call this decomposition the canonical orthant decomposition of $\mathbb{Z}^n$.

Example 2.3. Let $\mathbb{Z}_0^{(n)}$ be the sub-semigroup of $\mathbb{Z}^n$ generated by

$$\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\},$$

and $\mathbb{Z}_j^{(n)}$ be the sub-semigroup of $\mathbb{Z}^n$ generated by

$$\{(-1, \ldots, -1)\} \cup \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\} \backslash \{(\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0)\},$$
$$j = 1, 2, \ldots, n.$$

Then $\{\mathbb{Z}_0^{(n)}, \mathbb{Z}_1^{(n)}, \ldots, \mathbb{Z}_n^{(n)}\}$ is an orthant decomposition of $\mathbb{Z}^n$. For $n = 2$, we have

$$\mathbb{Z}_0^{(n)} = \{(a, b) | a \geqslant 0, b \geqslant 0, a, b \in \mathbb{Z}\},$$
$$\mathbb{Z}_1^{(n)} = \{(a, b) | a \leqslant 0, b \geqslant a, a, b \in \mathbb{Z}\},$$
$$\mathbb{Z}_2^{(n)} = \{(a, b) | b \leqslant 0, a \geqslant b, a, b \in \mathbb{Z}\}.$$

Definition 2.4. *Let $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. Then $a = (k_1, \ldots, k_m, l_1, \ldots, l_n)$ and $b = (r_1, \ldots, r_m, s_1, \ldots, s_n)$ of $\mathbb{N}^m \times \mathbb{Z}^n$ are called similar elements, if the n-tuples $(l_1, \ldots, l_n)$ and $(s_1, \ldots, s_n)$ are in the same orthant $\mathbb{Z}_j^{(n)}$ of $\mathbb{Z}^n$. In this case we also say a is similar to b.*

Definition 2.5. *Let $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. A total order "$\prec$" on $\mathbb{N}^m \times \mathbb{Z}^n$ is called a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ with respect to the decomposition, if the following conditions hold*:

(i) *$(0, \ldots, 0)$ is the smallest elements in $\mathbb{N}^m \times \mathbb{Z}^n$;*

(ii) *if $a \prec b$, then for any c similar to b, $a + c \prec b + c$, where $a, b, c \in \mathbb{N}^m \times \mathbb{Z}^n$.*

Remark. Def. 2.5 (ii) means that in every orthant the order just is a usual term order.

Example 2.6. Let $\{\mathbb{Z}_j^{(n)}, j = 1, \dots, 2^n\}$ be the canonical orthant decomposition of $\mathbb{Z}^n$ defined in Example 2.2. For every $a = (k_1, \dots, k_m, l_1, \dots, l_n) \in \mathbb{N}^m \times \mathbb{Z}^n$ let

$$|a| = k_1 + \cdots + k_m + |l_1| + \cdots + |l_n|.$$

For two elements $a = (k_1, \dots, k_m, l_1, \dots, l_n)$ and $b = (r_1, \dots, r_m, s_1, \dots, s_n)$ of $\mathbb{N}^m \times \mathbb{Z}^n$ define $a \prec b$ if and only if the $m + n + 1$-tuple $(|a|, k_1, \dots, k_m, l_1, \dots, l_n)$ is smaller than $(|b|, r_1, \dots, r_m, s_1, \dots, s_n)$ relative to the lexicographic order on $\mathbb{N}^{m+1} \times \mathbb{Z}^n$. Then "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$.

In fact, obviously $(0, \dots, 0)$ is the smallest elements in $\mathbb{N}^m \times \mathbb{Z}^n$. Now let $a \prec b$ and c be similar to b. Then $|a| \leqslant |b|$. We have

$$|a + c| \leqslant |a| + |c| \leqslant |b| + |c| = |b + c|. \tag{2.1}$$

The last equation holds because c is similar to b. If $|a + c| < |b + c|$, then $a + c \prec b + c$. If $|a + c| = |b + c|$, then $|a| = |b|$ must hold by (2.1). So the $m + n$-tuple $(k_1, \dots, k_m, l_1, \dots, l_n)$ is lexicographically smaller than $(r_1, \dots, r_m, s_1, \dots, s_n)$. Then $(k_1 + u_1, \dots, k_m + u_m, l_1 + v_1, \dots, l_n + v_n)$ is lexicographically smaller than $(r_1 + u_1, \dots, r_m + u_m, s_1 + v_1, \dots, s_n + v_n)$ for $c = (u_1, \dots, u_m, v_1, \dots, v_n)$ similar to b. In this case we also have $a + c \prec b + c$.

Example 2.7. Let the orthant decomposition of $\mathbb{Z}^n$ be as in Example 2.2. For every $a = (k_1, \dots, k_m, l_1, \dots, l_n) \in \mathbb{N}^m \times \mathbb{Z}^n$ let $|a|_1 = \sum_{j=1}^m k_j$, $|a|_2 = \sum_{j=1}^n |l_j|$. For two elements $a = (k_1, \dots, k_m, l_1, \dots, l_n)$ and $b = (r_1, \dots, r_m, s_1, \dots, s_n)$ of $\mathbb{N}^m \times \mathbb{Z}^n$ define $a \prec b$ if and only if the $m + 2n + 2$-tuple $(|a|_1, |a|_2, k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n)$ is lexicographically smaller than $(|b|_1, |b|_2, r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n)$. Then "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$.

First, we note that it is obvious that $(0, \dots, 0)$ is the smallest elements. Then, let $a \prec b$ and $c = (u_1, \dots, u_m, v_1, \dots, v_n)$ be similar to b. Because $|a|_1 \leqslant |b|_1$, so $|a + c|_1 \leqslant |b + c|_1$. But $|a + c|_1 < |b + c|_1$ would imply $a + c \prec b + c$, we can suppose $|a + c|_1 = |b + c|_1$. This would imply $|a|_1 = |b|_1$ and then $|a|_2 \leqslant |b|_2$. A relation similar to (2.1) would give $|a + c|_2 \leqslant |b + c|_2$. In the "$<$" case $a + c \prec b + c$ would hold.

Now suppose $|a + c|_1 = |b + c|_1$, $|a + c|_2 = |b + c|_2$. Then $|a|_1 = |b|_1$, $|a|_2 = |b|_2$. Note that for $j = 1, \dots, n$,

$$|l_j + v_j| \leqslant |l_j| + |v_j| \leqslant |s_j| + |v_j| = |s_j + v_j|.$$

So if $(k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n)$ is lexicographically smaller than $(r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n)$, then $a + c \prec b + c$.

Example 2.8. Let $\{\mathbb{Z}_j^{(n)}, j = 0, 1, \dots, n\}$ be the orthant decomposition of $\mathbb{Z}^n$ defined in Example 2.3. For every $a = (k_1, \dots, k_m, l_1, \dots, l_n) \in \mathbb{N}^m \times \mathbb{Z}^n$ let $\|a\| = -\min\{0, l_1, \dots, l_n\}$. For two elements $a = (k_1, \dots, k_m, l_1, \dots, l_n)$ and $b = (r_1, \dots, r_m, s_1, \dots, s_n)$ of $\mathbb{N}^m \times \mathbb{Z}^n$ define $a \prec b$ if and only if the $m + n + 1$-tuple $(\|a\|, k_1, \dots, k_m, l_1, \dots, l_n)$ is lexicographically smaller than $(\|b\|, r_1, \dots, r_m, s_1, \dots, s_n)$. Then "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$.

To prove this, note that $\mathbb{Z}_j^{(n)} = \{(i_1, \dots, i_n) \mid i_j \leqslant 0;\ i_k \geqslant i_j, k \neq j\}$, $j = 1, \dots, n$. It would imply $\min\{i_1, \dots, i_n\} = i_j$ when $(i_1, \dots, i_n) \in \mathbb{Z}_j^{(n)}$. Then for any $a, c \in \mathbb{N}^m \times \mathbb{Z}^n$ we have $\|a + c\| \leqslant \|a\| + \|c\|$. Equality holds if and only if that a and c are similar elements. Then it is clear that the "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ following the way as in Example 2.6.

In order to investigate Δ-σ-modules, we extend the notion of generalized term order to $\mathbb{N}^m \times \mathbb{Z}^n \times E$, where $E = \{e_1, \dots, e_q\}$ is a set of generators of a module.

Definition 2.9. *Let $\{\mathbb{Z}_j^{(n)}, j = 1, \dots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. Let $E = \{e_1, \dots, e_q\}$ be a set of q distinct elements. A total order $\prec$ on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is called a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to the decomposition, if the following conditions hold:*

(i) $(0, \dots, 0, e_i)$ *is the smallest element in* $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$, $e_i \in E$;

(ii) *if* $(a, e_i) \prec (b, e_j)$, *then for any c similar to b,* $(a + c, e_i) \prec (b + c, e_j)$, *where* $a, b, c \in \mathbb{N}^m \times \mathbb{Z}^n$, $e_i, e_j \in E$.

There are many ways to extend a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ to $\mathbb{N}^m \times \mathbb{Z}^n \times E$. Of course such an extended term order may also be defined directly. Some typical examples are shown below.

Example 2.10. Let the orthant decomposition of $\mathbb{Z}^n$ and the generalized term order "$\prec$" on $\mathbb{N}^m \times \mathbb{Z}^n$ be as in Example 2.7. Given a order "$\prec'$" in $E = \{e_1, \dots, e_q\}$, for two elements $(a, e_i) = (k_1, \dots, k_m, l_1, \dots, l_n, e_i)$ and $(b, e_j) = (r_1, \dots, r_m, s_1, \dots, s_n, e_j)$ of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ define

$$
\begin{aligned}
&(a, e_i) \prec_1 (b, e_j) \Longleftrightarrow a \prec b \quad \text{or} \quad (a = b \ \text{ and } \ e_i \prec' e_j);\\
&(a, e_i) \prec_2 (b, e_j) \Longleftrightarrow e_i \prec' e_j \quad \text{or} \quad (e_i = e_j \ \text{ and } \ a \prec b);\\
&(a, e_i) \prec_3 (b, e_j) \Longleftrightarrow (|a|_1, |a|_2, e_i, k_1, \dots, k_m, |l_1|, \dots, |l_n|, l_1, \dots, l_n)\\
&\qquad\qquad\qquad < (|b|_1, |b|_2, e_j, r_1, \dots, r_m, |s_1|, \dots, |s_n|, s_1, \dots, s_n) \ \text{ in lexicographic order.}
\end{aligned}
$$

Then "$\prec_1$", "$\prec_2$", "$\prec_3$" are all generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$.

"$\prec_1$" is called TOP extension of "$\prec$" and "$\prec_2$" is called POT extension of "$\prec$". "$\prec_3$" is a generalized term order defined directly.

Lemma 2.11. *Let $\{\mathbb{Z}_j^{(n)}, j = 1, \dots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$ and "$\prec$" be a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ with respect to the orthant decomposition. Suppose every orthant $\mathbb{Z}_j^{(n)}$ is isomorphic to $\mathbb{N}^n$ as a semigroup. Then every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n$ is finite. In particular, any subset of $\mathbb{N}^m \times \mathbb{Z}^n$ contains a smallest element.*

Proof. Let $a_1 \succ a_2 \succ a_3 \succ \cdots$ be a strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n$. Since there are finitely many orthants, without loss of generality we may assume that all a_j are similar elements which are in $\mathbb{N}^m \times \mathbb{Z}_i^{(n)}$ for a fixed i. By the condition of the Lemma, $\mathbb{N}^m \times \mathbb{Z}_i^{(n)}$ is isomorphic to $\mathbb{N}^{m+n}$ as a semigroup. Define order $\prec_1$ on $\mathbb{N}^{m+n}$ as $a \prec_1 b \Longleftrightarrow f^{-1}(a) \prec f^{-1}(b)$, where f is the isomorphic map from $\mathbb{N}^m \times \mathbb{Z}_i^{(n)}$ to $\mathbb{N}^{m+n}$ and $\prec$ is the generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$. Since $\prec$ is a term order on $\mathbb{N}^m \times \mathbb{Z}_i^{(n)}$, it follows that $\prec_1$ is a term order on $\mathbb{N}^{m+n}$. Then the assertion of the Lemma follows from the well-order property of term order on $\mathbb{N}^{m+n}$.

Remark. In Lemma 2.11 the condition "every orthant $\mathbb{Z}_j^{(n)}$ is isomorphic to $\mathbb{N}^n$ as a semigroup" is necessary. From Definition 2.1 we can not deduce the condition. Also, there are counterexamples illustrate that the Lemma can not holds without the condition.

Lemma 2.11 means that an algorithm based on a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ may terminate after finite steps. To deal with difference-differential modules we have following corollary.

Corollary 2.12. *Given an orthant decomposition of $\mathbb{Z}^n$ and a generalized term order "$\prec$" on $\mathbb{N}^m \times \mathbb{Z}^n \times E$, every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is finite. In particular, any subset of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ contains a smallest element.*

Proof. Let $a_1 \succ a_2 \succ a_3 \succ \cdots$ be a strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n \times E$. Since E is a finite set, we may suppose that all a_j are in $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$ for an i. Then Lemma 2.11 shows that the conclusion holds.

3 Gröbner bases in finitely generated difference-differential-modules

In classical Gröbner basis theory, the concept of reduction of polynomials is essential. To describe reduction in a difference-differential-module, we first investigate some basic multiplication and division properties in the module.

Let $\{\mathbb{Z}_j^{(n)}, j = 1, \dots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$ and "$\prec$" be a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n$ with respect to the orthant decomposition. Let Λ be the semi-group introduced in Section 1 in which the elements are of the form (1.1). Since Λ is isomorphic to $\mathbb{N}^m \times \mathbb{Z}^n$ as a semigroup, the "$\prec$" would define an order on Λ. We also call it a generalized term order on Λ.

Let K be a Δ-σ-field and D be the ring of Δ-σ-operators over K, and let F be a finitely generated free D-module (i.e. a finitely generated free difference-differential-module) with a set of free generators $E = \{e_1, \dots, e_q\}$. Then F can be considered as a K-vector space generated by the set of all elements of the form λe_i ($i = 1, \dots, q$, where $\lambda \in \Lambda$). This set will be denoted by ΛE and its elements will be called "terms" of F. In particular the elements of Λ will be called "terms" of D. If "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times B$ then "$\prec$" would define a generalized term order on ΛE.

It is clear that every element $f \in F$ has a unique representation as a linear combination of terms:

$$f = a_1 \lambda_1 e_{j_1} + \cdots + a_d \lambda_d e_{j_d} \tag{3.1}$$

for some nonzero elements $a_i \in R$ ($i = 1, \dots, d$) and some distinct elements $\lambda_1 e_{j_1}, \dots, \lambda_d e_{j_d} \in \Lambda E$.

If a term λe_j appears in the form (3.1) of f, then it is called a term of f. If $(k_1, \dots, k_m, l_1, \dots, l_n)$ and $(r_1, \dots, r_m, s_1, \dots, s_n)$ are similar elements in $\mathbb{N}^m \times \mathbb{Z}^n$ then the two terms $\lambda_1 = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n}$ and $\lambda_2 = \delta_1^{r_1} \cdots \delta_m^{r_m} \alpha_1^{s_1} \cdots \alpha_n^{s_n}$ of D are called similar. If $\lambda_1, \lambda_2 \in \Lambda$ are similar, then the two terms $u = \lambda_1 e_i, v = \lambda_2 e_j \in \Lambda E$ are called similar.

Definition 3.1. *Let "$\prec$" be a generalized term order on ΛE, $f \in F$ be of the form* (3.1). *Then $\mathrm{lt}(f) = \max_{\prec}\{\lambda_i e_{j_i} | i = 1, \dots, d\}$ is called the leading term of f. If $\lambda_i e_{j_i} = \mathrm{lt}(f)$, then $\mathrm{lc}(f) = a_i$ is called the leading coefficient of f.*

Note that in the case of that "$\prec$" is a generalized term order, in general the equation $\lambda \mathrm{lt}(f) = \mathrm{lt}(\lambda f)$ is not true unless the leading term $\mathrm{lt}(f) = \eta e_i$ of f is such that η is similar to λ.

Now we are going to construct the division algorithm in the difference-differential module F. First we give some lemmas to describe the multiple properties in difference-differential modules. In what follows we always assume that an orthant decomposition of $\mathbb{Z}^n$ is given and a generalized term order is with respect to the decomposition.

Definition 3.2. *Let λ be the form of* (1.1)*. Then the subset Λ_j of Λ,*

$$\Lambda_j = \{\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n} | (l_1, \ldots, l_n) \in \mathbb{Z}_j^{(n)}\},$$

where $\mathbb{Z}_j^{(n)}$ is the j-th orthant of the decomposition of $\mathbb{Z}^n$, is called j-th orthant of Λ. Let F be a finitely generated free D-module and ΛE be the set of terms of F. Then $\Lambda_j E = \{\lambda e_i | \lambda \in \Lambda_j,\ e_i \in E\}$ is called j-th orthant of ΛE.

Obviously, two elements in Λ or ΛE are similar if and only if they are in same orthant. So from Def. 2.5, if "$\prec$" is a generalized term order on Λ and $\xi \prec \lambda$, then $\eta\xi \prec \eta\lambda$ holds for any η in the same orthant as λ.

Lemma 3.3. *Let $\lambda \in \Lambda$ and $a \in K$, "$\prec$" be a generalized term order on $\Lambda \subseteq D$. Then $\lambda a = a'\lambda + \xi$, where $a' = \alpha(a)$ for some $\alpha \in \Gamma$ (see* (1.1)*), and if $a \neq 0$ then $a' \neq 0$; $\xi \in D$ with $\mathrm{lt}(\xi) \prec \lambda$ and all terms of ξ are similar to λ.*

Proof. Let $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n}$ as (1.1). Denote $\alpha_1^{l_1} \cdots \alpha_n^{l_n}$ by α. Then by the fundamental relations (1.3) we have

$$\lambda a = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha(a) \alpha = \delta_1^{k_1} \cdots \delta_m^{k_m} a' \alpha = (a' \delta_1^{k_1} \cdots \delta_m^{k_m} + \eta)\alpha = a' \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha + \eta\alpha,$$

where $\eta \in R[\Delta]$, $a' = \alpha(a)$. Because $\alpha_j \in \sigma$, $j = 1, \ldots, n$, are invertible, we have $a' \neq 0$ if $a \neq 0$.

If $\mathrm{lt}(\eta) = \delta_1^{k_1'} \cdots \delta_m^{k_m'}$, then it is obvious that $(k_1, \ldots, k_m) \in \{(k_1', \ldots, k_m') + \mathbb{N}^m\}$ from (1.3). This means that $\mathrm{lt}(\eta) \prec \delta_1^{k_1} \cdots \delta_m^{k_m}$. Furthermore, since $\delta\alpha$ and $\eta\alpha$ are always similar for α in σ and δ, η in Δ, we see that all terms of $\xi = \eta\alpha$ are similar to λ. Since $\delta_1^{k_1} \cdots \delta_m^{k_m}$ is in every orthant of Λ, it follows that $\mathrm{lt}(\xi) = \mathrm{lt}(\eta\alpha) \prec \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha = \lambda$.

In general $\mathrm{lt}(\lambda f) = \lambda \mathrm{lt}(f)$ is not true. But we have the following

Lemma 3.4. *Let F be a finitely generated free D-module and $0 \neq f \in F$. Then the following assertions hold*:

(i) *If $\lambda \in \Lambda$, then $\mathrm{lt}(\lambda f) = \max_\prec\{\lambda u_i\}$ where u_i are terms of f and then $\mathrm{lt}(\lambda f) = \lambda u$ for a unique term u of f.*

(ii) *If $\mathrm{lt}(f) \in \Lambda_j e$ then for any $\lambda \in \Lambda_j$, $\mathrm{lt}(\lambda f) = \lambda \mathrm{lt}(f) \in \Lambda_j E$.*

Proof. (i) Suppose that $f = \sum_{i=1}^d a_i \lambda_i e_{j_i}$ as (3.1) and $\lambda \in \Lambda$, then by Lemma 3.3 we have

$$\lambda f = \sum_{i=1}^d \lambda a_i \lambda_i e_{j_i} = \sum_{i=1}^d (a_i' \lambda + \xi_i) \lambda_i e_{j_i}, \tag{3.2}$$

where $\xi_i \in D$ with $\mathrm{lt}(\xi_i) \prec \lambda$. Note that $\xi_i = \eta_i \alpha$ in the proof of Lemma 3.3, where $\eta_i \in R[\Delta]$, $\alpha \in \Gamma$. So every term of ξ_i is the form of $\sigma_i \alpha$ with $\sigma_i \in \Theta$ (see (1.1)) and $\alpha = \alpha_1^{l_1} \cdots \alpha_n^{l_n}$ such that $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n}$ as in the proof of Lemma 3.3. It follows that

$$\sigma_i \alpha \lambda_i e_{j_i} \prec \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha \lambda_i e_{j_i} = \lambda \lambda_i e_{j_i}. \tag{3.3}$$

So $\mathrm{lt}(\lambda f) \in \{\lambda \lambda_i e_{j_i}\}$. If $\lambda \lambda_i e_{j_i} = \lambda \lambda_i' e_{j_i}'$ then $e_{j_i} = e_{j_i}'$ and $\lambda_i = \lambda_i'$. Therefore $\mathrm{lt}(\lambda f) = \max_\prec\{\lambda \lambda_i e_{j_i} | i = 1, \ldots, d\} = \lambda u$ for a unique term u of f.

(ii) Suppose that f is as above and $\mathrm{lt}(f) = \lambda_1 e_{j_1} \in \Lambda_j E$. If $\lambda \in \Lambda_j$, then in (3.2) we have $\lambda_i e_{j_i} \prec \lambda_1 e_{j_1}$. Then λ similar to λ_1 implies

$$\lambda\lambda_i e_{j_i} \prec \lambda\lambda_1 e_{j_1}. \tag{3.4}$$

From (3.3) and (3.4) we conclude that $\mathrm{lt}(\lambda f) = \lambda \mathrm{lt}(f) \in \Lambda_j E$.

Lemma 3.5. *Let F be a finitely generated free D-module and $0 \neq f \in F$. Then for each j, there exists some $\lambda \in \Lambda$ and a term u_j of f such that $\mathrm{lt}(\lambda f) = \lambda u_j \in \Lambda_j E$. Furthermore, the term u_j of f is unique: if $\mathrm{lt}(\lambda_1 f) = \lambda_1 u_{j_1} \in \Lambda_j E$ and $\mathrm{lt}(\lambda_2 f) = \lambda_2 u_{j_2} \in \Lambda_j E$ then $u_{j_1} = u_{j_2}$. We will write $\mathrm{lt}_j(f)$ for this term u_j.*

Proof. Let f be the form of (3.1) and then $\{\lambda_i e_{j_i} | i = 1, \ldots, d\}$ be the set of terms of f. Let $\lambda_i = \delta^{s_i}\alpha^{t_i}$, $s_i \in \mathbb{N}^m$ and $t_i \in \mathbb{Z}^n$. By Def. 2.1 (iii), the group generated by $\mathbb{Z}_j^{(n)}$ is $\mathbb{Z}^n$. Therefore there exist $u_i,\ v_i \in \mathbb{Z}_j^{(n)}$ such that $u_i - v_i = t_i$. This means that $\alpha^{v_i}\lambda_i = \delta^{s_i}\alpha^{u_i} \in \Lambda_j$. Put $\zeta_i = \alpha^{v_i}$ and $\lambda = \prod_{i=1}^d \zeta_i$, then $\lambda \cdot \lambda_i \in \Lambda_j$ holds for all $i = 1, \ldots, d$. Now we have

$$\lambda f = \sum_{i=1}^{d} \lambda a_i \lambda_i e_{j_i} = \sum_{i=1}^{d} (a_i' \lambda + \xi_i)\lambda_i e_{j_i}$$

from Lemma 3.3. Because there is no δ factor in λ, $\xi_i = 0$ from the proof of Lemma 3.3. Then all terms of λf are in $\Lambda_j E$ and $\mathrm{lt}(\lambda f) \in \Lambda_j E$.

Now suppose that there are terms $u,\ v$ of f such that $\mathrm{lt}(\lambda f) = \lambda u \in \Lambda_j E$, $\mathrm{lt}(\eta f) = \eta v \in \Lambda_j E$. For $\lambda,\ \eta \in \Lambda$ the above proof shows that there is $\zeta \in \Lambda_j$ such that $\zeta\lambda,\ \zeta\eta \in \Lambda_j$. Since $\lambda v \preceq \lambda u$, $\eta u \preceq \eta v$ then $\zeta\lambda v \preceq \zeta\lambda u$, $\zeta\eta u \prec \zeta\eta v$ because $\zeta \in \Lambda_j$. Furthermore, this would imply $(\zeta\eta)\zeta\lambda v \preceq (\zeta\eta)\zeta\lambda u$, $(\zeta\lambda)\zeta\eta u \preceq (\zeta\lambda)\zeta\eta v$ because $\zeta\lambda,\ \zeta\eta \in \Lambda_j$. Then $\zeta\eta\zeta\lambda v = \zeta\eta\zeta\lambda u$ and then $u = v$.

Denote the term $u = v$ of f by $\mathrm{lt}_j(f)$, then for any $\lambda \in \Lambda$ such that $\mathrm{lt}(\lambda f) \in \Lambda_j E$, $\mathrm{lt}(\lambda f) = \lambda \mathrm{lt}_j(f)$.

Remark. Lemma 3.5 asserts that, either $\lambda \in \Lambda_j$ or $\lambda \notin \Lambda_j$, we have $\mathrm{lt}(\lambda f) = \lambda \mathrm{lt}_j(f)$ for a unique $u = \mathrm{lt}_j(f)$ as long as $\mathrm{lt}(\lambda f) \in \Lambda_j E$.

For instance, let $f = \alpha_1^{-2}\alpha_2^3 + \alpha_1^5\alpha_2^{-2}$ (the generalized term order as in Example 2.6 and $\alpha_1, \alpha_2 \in \sigma$). $\lambda = \alpha_1^{-1}\alpha_2 \in \Lambda_2$, $\eta = \alpha_1^{-3}\alpha_2^{-1} \in \Lambda_3$. Then $\lambda f = \alpha_1^{-3}\alpha_2^4 + \alpha_1^4\alpha_2^{-1}$, $\eta f = \alpha_1^{-5}\alpha_2^2 + \alpha_1^2\alpha_2^{-3}$ and $\mathrm{lt}(\lambda f) = \alpha_1^{-3}\alpha_2^4 \in \Lambda_2 = \lambda\alpha_1^{-2}\alpha_2^3$, $\mathrm{lt}(\eta f) = \alpha_1^{-5}\alpha_2^2 \in \Lambda_2 = \eta\alpha_1^{-2}\alpha_2^3$. Then $\mathrm{lt}_2(f) = \alpha_1^{-2}\alpha_2^3$. Note that $\mathrm{lt}(f) = \alpha_1^5\alpha_2^{-2} \neq \mathrm{lt}_2(f)$.

If $h = \sum_{i\in I} b_i\lambda_i \in D$, $f = \sum_{j\in J} c_j u_j \in F$, then $hf = \sum_{i\in I, j\in J} b_i c_j \lambda_i u_j$. Since some of $\lambda_i u_j$ may be equal and vanish in terms of hf, it would be problematic if $\mathrm{lt}(hf) \prec \lambda_i u_j$ might occur for some λ_i and u_j. The following proposition asserts that this undesirable situation cannot occur.

Proposition 3.6. *Let $0 \neq f \in F$, $0 \neq h \in D$. Then $\mathrm{lt}(hf) = \max_{\prec}\{\lambda_i u_k\}$ where λ_i are terms of h and u_k are terms of f. Therefore $\mathrm{lt}(hf) = \lambda u$ for a unique term λ of h and a unique term u of f.*

Proof. Let $h = \sum_{i\in I} b_i\lambda_i$ where I is a finite set and λ_i, $i \in I$, are distinct elements in Λ. Then $hf = \sum_{i\in I} b_i\lambda_i f$. By Lemma 3.4 (i), there is a unique term u_{k_i} of f such that $\mathrm{lt}(\lambda_i f) = \lambda_i u_{k_i} \succeq \lambda_i u_k$ for all terms u_k of f. Also we have that $\mathrm{lt}(\lambda_i f) = \lambda_i u_{k_i}$, $i \in I$, are distinct: if $\mathrm{lt}(\lambda_{i_1}f) = \mathrm{lt}(\lambda_{i_2}f)$ then they must be in a same $\Lambda_j E$. It follows from Lemma 3.5 that there is a unique term $\mathrm{lt}_j(f)$ of f such that

$$\mathrm{lt}(\lambda_{i_1}f) = \lambda_{i_1}\mathrm{lt}_j(f) = \mathrm{lt}(\lambda_{i_2}f) = \lambda_{i_2}\mathrm{lt}_j(f) \in \Lambda_j E.$$

Therefore $\lambda_{i_1} = \lambda_{i_2}$. Since λ_i, $i \in I$ are distinct, this is impossible. So we have $\mathrm{lt}(hf) \in \{\lambda_i u_{k_i}\}_{i\in I} = \{\mathrm{lt}(\lambda_i f)\}_{i\in I}$. This means that there is a unique i_0 such that

$$\mathrm{lt}(hf) = \mathrm{lt}(\lambda_{i_0}f) = \lambda_{i_0}u_{k_{i_0}} \succeq \lambda_i u_k \tag{3.5}$$

for all terms λ_i of h and all terms u_k of f.

Theorem 3.7. *Let $f_1, \dots, f_p \in F\backslash\{0\}$. Then every $g \in F$ can be represented as*

$$g = h_1 f_1 + \cdots + h_p f_p + r \tag{3.6}$$

for some elements $h_1, \dots, h_p \in D$ and $r \in F$ such that

(i) $h_i = 0$ *or* $\mathrm{lt}(h_i f_i) \preceq \mathrm{lt}(g)$, $i = 1, \dots, p$; (*By Proposition* 3.1 *this means that* $\lambda u \preceq \mathrm{lt}(g)$ *for all terms λ of h_i and all terms u of f_i.*)

(ii) $r = 0$ *or* $\mathrm{lt}(r) \preceq \mathrm{lt}(g)$ *such that* $\mathrm{lt}(r) \notin \{\mathrm{lt}(\lambda f_i) | \lambda \in \Lambda, i = 1, \dots, p\}$.

Proof. The elements $h_1, \dots, h_p \in D$ and $r \in F$ can be computed as follows:

First set $r = g$ and $h_i = 0$, $i = 1, \dots, p$.

Suppose $r \neq 0$, i.e. $r = \mathrm{lc}(r)\mathrm{lt}(r) + r'$ and $\mathrm{lt}(r) = \mathrm{lt}(\lambda_i f_i)$ for some f_i and an element $\lambda_i \in \Lambda$. Then $\lambda_i f_i = c_i \mathrm{lt}(\lambda_i f_i) + \xi_i$, where $c_i = \mathrm{lc}(\lambda_i f_i)$ and $\mathrm{lt}(\xi_i) \prec \mathrm{lt}(\lambda_i f_i)$. Therefore

$$r = \mathrm{lc}(r)\mathrm{lt}(r) + r' = \mathrm{lc}(r)\mathrm{lt}(\lambda_i f_i) + r' = \frac{\mathrm{lc}(r)}{c_i}(\lambda_i f_i - \xi_i) + r'.$$

Put $b_i = \frac{\mathrm{lc}(r)}{c_i}$ and $r_i = \frac{\mathrm{lc}(r)}{c_i}(-\xi_i) + r'$. Then $r = b_i\lambda_i f_i + r_i$. Now we may replace r by r_i and h_i by $h_i + b_i\lambda_i$. Since in each step we have $\mathrm{lt}(r_i) \prec \mathrm{lt}(\lambda_i f_i) \preceq \mathrm{lt}(r) \preceq \mathrm{lt}(g)$, by Corollary 2.12, the algorithm above terminates after finitely many iterations.

Remark. Since $\eta\mathrm{lt}(\lambda_i f_i) = \mathrm{lt}(\eta\lambda_i f_i) = \mathrm{lt}(\lambda_i' f_i)$ for any η similar to $\mathrm{lt}(\lambda_i f_i)$ from Lemma 3.4(ii), the condition (ii) in Theorem 3.7 means that $r = 0$ or $\mathrm{lt}(r) \preceq \mathrm{lt}(g)$ such that $\mathrm{lt}(r)$ is not in $\Lambda_j\mathrm{lt}(\lambda_i f_i)$ if $\mathrm{lt}(\lambda_i f_i) \in \Lambda_j E$.

Definition 3.8. *Let $f_1, \dots, f_p \in F\backslash\{0\}$, $g \in F$. Suppose that the equality* (3.6) *holds and the conditions* (i), (ii) *in Theorem* 3.7 *are satisfied. If $r \neq g$ we say g can be reduced by $\{f_1, \dots, f_p\}$ to r. In this case we have* $\mathrm{lt}(r) \prec \mathrm{lt}(g)$ *by the proof of Theorem* 3.7. *In the case of $r = g$ and $h_i = 0$, $i = 1, \dots, p$, we say that g is reduced with respect to $\{f_1, \dots, f_p\}$.*

The following example illustrates the reason for the condition (ii) in Theorem 3.7.

Example 3.9. Let $K = \mathbb{Q}(x_1, x_2)$, $D = K[\delta_1, \delta_2, \alpha, \alpha^{-1}]$, where δ_1, δ_2 are the partial derivative by x_1, x_2 respectively, and α is an automorphism of K. Choose generalized term order on $\mathbb{N}^2 \times \mathbb{Z}$ as in Example 2.6, i.e.

$$u = \delta_1^{k_1}\delta_2^{k_2}\alpha^l \prec v = \delta_1^{r_1}\delta_2^{r_2}\alpha^s \Longleftrightarrow (|u|, k_1, k_2, l) < (|v|, r_1, r_2, s) \text{ in lexicographic order,}$$

where $|u| = k_1 + k_2 + |l|$.

Let $g = \delta_1\alpha^{-2} + \delta_2\alpha^2$, $f = \delta_1\alpha^{-1} + \alpha$. Then

$$g = \delta_1\alpha^{-2} + \delta_2\alpha^2 = \alpha^{-1}(\delta_1\alpha^{-1} + \alpha) + (\delta_2\alpha^2 - 1) = \alpha^{-1}f + r_1.$$

Although $\mathrm{lt}(r_1) = \delta_2\alpha^2$ is not any multiple of $\mathrm{lt}(f) = \delta_1\alpha^{-1}$, we can find $\lambda = \delta_2\alpha$ such that $\mathrm{lt}(r_1) = \mathrm{lt}(\lambda f) = \mathrm{lt}(\delta_1\delta_2 + \delta_2\alpha^2)$. Therefore

$$g = \alpha^{-1}f + \delta_2\alpha f + (-\delta_1\delta_2 - 1) = (\alpha^{-1} + \delta_2\alpha)f + r_2.$$

Now r_2 satisfies the condition (ii) in Theorem 3.7. Then g is reduced by f to r_2.

The concept of classical Gröbner bases in commutative polynomial algebra can be defined in several ways. The essential point is: G is a Gröbner basis of W if every $f \in W$ can be reduced to 0 by G. This means that $\mathrm{lt}(f) = \lambda\mathrm{lt}(g_j) = \mathrm{lt}(\lambda g_j)$ for some $g_j \in G$. We can go along this way to define difference-differential Gröbner bases.

Definition 3.10. *Let W be a submodule of the finitely generated free D-module F and $\prec$ be a generalized term order on ΛE. $G = \{g_1, \dots, g_p\} \in W\backslash\{0\}$. Then G is called a Gröbner basis of W (with respect to the generalized term order $\prec$) if for any $f \in W\backslash\{0\}$, $\mathrm{lt}(f) = \mathrm{lt}(\lambda g_i)$ for some $\lambda \in \Lambda$, $g_i \in G$. If every element of G is reduced with respect to other element of G, then G is called a reduced Gröbner basis of W.*

Remark. Using $\mathrm{lt}_j(g)$ which is introduced in Lemma 3.5, Definition 3.10 is equivalent to: G is a Gröbner basis of W iff for any $f \in W\backslash\{0\}$, if $\mathrm{lt}(f) \in \Lambda_j E$, then $\mathrm{lt}(f) = \lambda\mathrm{lt}_j(g_i)$ for some $\lambda \in \Lambda$, $g_i \in G$. (Note that $\lambda\mathrm{lt}_j(g_i) = \mathrm{lt}(\lambda g_i)$ by Lemma 3.5. We see that the style of Definition 3.10 is simple and clear.)

Proposition 3.11. *Let G be a finite subset of $W\backslash\{0\}$. The following assertions hold*:

(i) *G is a Gröbner basis of W if and only if every $f \in W$ can be reduced by G to* 0. *So a Gröbner basis of W generates the D-module W.*

(ii) *If G is a Gröbner basis of W, $f \in F$, then $f \in W$ if and only if f can be reduced by G to* 0.

(iii) *If G is a Gröbner basis of W, then $f \in W$ is reduced with respect to G if and only if* $f = 0$.

Proof. (i) If G is a Gröbner basis of W, $f \in W$, then from Theorem 3.7 f can be reduced by G to r with $\mathrm{lt}(r)$ does not equal any $\mathrm{lt}(\lambda g)$, $\lambda \in \Lambda$, $g \in G$. If $r \neq 0$ then $r \in W$, therefore $\mathrm{lt}(r) = \mathrm{lt}(\lambda g)$ for some $g \in G$ and some $\lambda \in \Lambda$,which is a contradiction.

If every $f \in W$ can be reduced by G to 0, then $f = \sum_{g\in G} h_g g$. By Proposition 3.6, there is a $g \in G$ such that $\mathrm{lt}(f) = \max_{g\in G}\{\mathrm{lt}(h_g g)\} = \lambda u$ for a term λ of h_g and a term u of g. Then $\mathrm{lt}(f) = \mathrm{lt}(\lambda g)$. By Definition 3.10, G is a Gröbner basis of W.

(ii) and (iii) follow easily from Theorem 3.7 and Definition 3.10.

Example 3.12. If W is generated by just one element $g \in F\backslash\{0\}$, then any finite subset G of $W\backslash\{0\}$ containing g is a Gröbner basis of W. In fact, $0 \neq f \in W$ implies $f = hg$ for some $h \in D\backslash\{0\}$. By Proposition 3.6, $\mathrm{lt}(f) = \lambda u = \max_{\prec}\{\lambda_i u_k\}$ for a term λ of h and a term u of g. Then $\mathrm{lt}(f) = \mathrm{lt}(\lambda g)$. By Definition 3.10, G is a Gröbner basis of W.

Below we will consider the Buchberger's algorithm for computing a Gröbner basis of a submodule W of F. This requires a suitable definition of the concept of S-polynomial. Since there are many orthants we have to compute S-polynomials in every orthant.

Definition 3.13. *Let F be a finitely generated free D-module and $f,\ g \in F\backslash\{0\}$. For every Λ_j let $V(j,f,g)$ be a finite system of generators (which are terms) of the $K[\Lambda_j]$-module ${}_{K[\Lambda_j]}\langle \mathrm{lt}(\lambda f) \in \Lambda_j E | \lambda \in \Lambda\rangle \bigcap_{K[\Lambda_j]}\langle \mathrm{lt}(\eta g) \in \Lambda_j E | \eta \in \Lambda\rangle$. Then for every generator $v \in V(j,f,g)$*

$$S(j,f,g,v) = \frac{v}{\mathrm{lt}_j(f)}\frac{f}{\mathrm{llc}_j(f)} - \frac{v}{\mathrm{lt}_j(g)}\frac{g}{\mathrm{lc}_j(g)}$$

is called an S-polynomial of f and g with respect to j and v.

The $K[\Lambda_j]$-module considered in Definition 3.13 is a "monomial module", i.e. it is generated by elements containing only one term. Such a module always has a finite "monomial basis", i.e. every basis element contains only one term. In the following we assume that $V(j,f,g)$ is such a finite monomial basis.

The computation of $V(j,f,g)$ is involved in the generalized term order on ΛE. Pauer and Unterkircher[8] researched $V(j,f,g)$ in commutative Laurent polynomial rings and gave algorithm for some important cases. Their results are still valid for difference-differential modules.

Example 3.14. Let $F = D = K[\delta_1,\delta_2,\alpha_1,\alpha_1^{-1},\alpha_2,\alpha_2^{-1}]$ and $K = \mathbb{Q}(x_1,x_2)$. Where δ_1, δ_2 are the partial derivative by x_1, x_2 respectively, and α_1, α_2 are two automorphism on K. Choose the generalized term order on $\mathbb{N}^2 \times \mathbb{Z}^2$ as in Example 2.8, i.e.

$$\begin{aligned} &u = \delta_1^{k_1}\delta_2^{k_2}\alpha_1^{l_1}\alpha_2^{l_2} \prec v = \delta_1^{r_1}\delta_2^{r_2}\alpha_1^{s_1}\alpha_2^{s_2} \\ &\quad\Longleftrightarrow (\|u\|,k_1,k_2,l_1,l_2) < (\|v\|,r_1,r_2,s_1,s_2) \ \text{ in lexicographic order,}\end{aligned}$$

where $\|u\| = -\min(0,l_1,l_2)$.

Let $f = \alpha_1^{-2} - \delta_2$, $g = \delta_1 + \alpha_2^4$. Note that the orthants of Λ are $\Lambda_0,\Lambda_1,\Lambda_2$ described in Example 2.3 for $n = 2$. Then we obtain that

$$\{\lambda \in \Lambda | \mathrm{lt}(\lambda f) \in \Lambda_0\} = \Lambda_0\alpha_1^2, \quad \{\eta \in \Lambda | \mathrm{lt}(\eta g) \in \Lambda_0\} = \Lambda_0;$$

and

$$\{\mathrm{lt}(\lambda f) \in \Lambda_0 | \lambda \in \Lambda\} = \Lambda_0\delta_2\alpha_1^2, \quad \{\mathrm{lt}(\eta g) \in \Lambda_0 | \eta \in \Lambda\} = \Lambda_0\delta_1.$$

Therefore $V(0,f,g) = \{v_0\} = \{\delta_1\delta_2\alpha_1^2\}$ and by Definition 3.13,

$$S(0,f,g,v_0) = \delta_1\alpha_1^2 f + \delta_2\alpha_1^2 g = \delta_1 + \delta_2\alpha_1^2\alpha_2^4.$$

Similarly we have

$$\begin{aligned} &\{\lambda \in \Lambda | \mathrm{lt}(\lambda f) \in \Lambda_1\} = \Lambda_1\alpha_1, \quad \{\eta \in \Lambda | \mathrm{lt}(\eta g) \in \Lambda_1\} = \Lambda_1; \\ &\{\mathrm{lt}(\lambda f) \in \Lambda_1 \mid \lambda \in \Lambda\} = \Lambda_1\alpha_1^{-1}, \quad \{\mathrm{lt}(\eta g) \in \Lambda_1 | \eta \in \Lambda\} = \Lambda_1\delta_1.\end{aligned}$$

So $V(1,f,g) = \{v_1\} = \{\delta_1\alpha_1^{-1}\}$ and $S(1,f,g,v_1) = \delta_1\alpha_1 f - \alpha_1^{-1} g = -\delta_1\delta_2\alpha_1 - \alpha_1^{-1}\alpha_2^4$. Finally,

$$\begin{aligned} &\{\lambda \in \Lambda | \mathrm{lt}(\lambda f) \in \Lambda_2\} = \Lambda_2\alpha_1^2, \quad \{\eta \in \Lambda | \mathrm{lt}(\eta g) \in \Lambda_2\} = \Lambda_2; \\ &\{\mathrm{lt}(\lambda f) \in \Lambda_2 | \lambda \in \Lambda\} = \Lambda_2\delta_2\alpha_1^2, \quad \{\mathrm{lt}(\eta g) \in \Lambda_2 | \eta \in \Lambda\} = \Lambda_2\delta_1.\end{aligned}$$

So $V(2,f,g) = \{v_2\} = \{\delta_1\delta_2\alpha_1^2\}$ and $S(2,f,g,v_2) = \delta_1\alpha_1^2 f + \delta_2\alpha_1^2 g = \delta_1 + \delta_2\alpha_1^2\alpha_2^4$.

For the proof of Theorem 3.17 we need the following lemmas:

Lemma 3.15. *Let $\{r_1, \ldots, r_l\} \subseteq F$ and $\{a_1, \ldots, a_l\} \subseteq K$. If $\sum_{j=1}^{l} a_j = 0$, then $\sum_{j=1}^{l} a_j r_j = \sum_{j=1}^{l-1} b_j(r_j - r_{j+1})$ for some $b_j \in K$.*

Proof. Obviously

$$\sum_{j=1}^{l} a_j r_j = a_1(r_1 - r_2) + (a_1 + a_2)(r_2 - r_3) + (a_1 + a_2 + a_3)(r_3 - r_4) + \cdots + (a_1 + a_2 + \cdots + a_{l-1})(r_{l-1} - r_l) + (a_1 + a_2 + \cdots + a_l)r_l.$$

Since $a_1 + a_2 + \cdots + a_l = 0$ it follows that $\sum_{j=1}^{l} a_j r_j = \sum_{j=1}^{l-1} b_j(r_j - r_{j+1})$ for some $b_j \in K$.

Lemma 3.16. *Let $g_i,\ g_k \in F$ and $\mathrm{lt}(\lambda g_i) = \mathrm{lt}(\eta g_k) = u \in \Lambda_j E$, where $\lambda,\ \eta \in \Lambda$. Then there exist $\zeta \in \Lambda_j$ and $v \in V(j, g_i, g_k)$ defined in Definition 3.13, such that $u = \zeta v$. Furthermore, if G is a finite subset of $F\backslash\{0\}$ and the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G then*

$$\zeta S(j, g_i, g_k, v) = \frac{u}{\mathrm{lt}_j(g_i)} \frac{g_i}{\mathrm{lc}_j(g_i)} - \frac{u}{\mathrm{lt}_j(g_k)} \frac{g_k}{\mathrm{lc}_j(g_k)} = \sum_{g \in G} h_g g$$

with $\mathrm{lt}(h_g g) \prec u$ for $g \in G$.

Proof. Suppose $V(j, g_i, g_k) = \{v_1, \ldots, v_l\}$. Then $u = \sum_\mu p_\mu v_\mu$, where $p_\mu \in K[\Lambda_j]$. Since $p_\mu = \sum_\nu a_{\mu\nu}\lambda_{\mu\nu}$, where $a_{\mu\nu} \in R$ and $\lambda_{\mu\nu} \in \Lambda_j$, it follows that $u = \sum_{\mu,\nu} a_{\mu\nu}(\lambda_{\mu\nu} v_\mu)$.

Note that u and $\lambda_{\mu\nu} v_\mu$ are terms in $\Lambda_j E$ and we can rewrite the right of the equation such that the terms $\lambda_{\mu\nu} v_\mu$ are distinct. Then we see that there is a unique $a_{\mu\nu} = 1$ and others are zero. Then $u = \zeta v$ for a $\zeta \in \Lambda_j$ and $v \in V(j, g_i, g_k)$.

Now if $S(j, g_i, g_k, v)$ can be reduced to 0 by G then $S(j, g_i, g_k, v) = \sum_{g \in G} h'_g g$ and $\mathrm{lt}(h'_g g) \preceq \mathrm{lt}(S(j, g_i, g_k, v)) \prec v$ for $g \in G$. Therefore $\zeta S(j, g_i, g_k, v) = \sum_{g \in G}(\zeta h'_g)g = \sum_{g \in G} h_g g$, where $h_g = \zeta h'_g$. By Lemma 3.4 (i), $\mathrm{lt}(\zeta h'_g g) = \zeta w$ for a term w of $h'_g g$. Then $\mathrm{lt}(h_g g) = \mathrm{lt}(\zeta h'_g g) = \zeta w$. Therefore $w \preceq \mathrm{lt}(h'_g g) \prec v$ and $\zeta \in \Lambda_j$ imply that $\zeta w \prec \zeta v = u$.

Theorem 3.17 (Generalized Buchberger Theorem)**.** *Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F\backslash\{0\}$ and W be the submodule in F generated by G. Then G is a Gröbner basis of W if and only if for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G.*

Proof. If G is a Gröbner basis of W, since $S(j, g_i, g_k, v)$ is an element of W, then it follows from Proposition 3.11 that $S(j, g_i, g_k, v)$ can be reduced to 0 by G.

Now let G be a finite subset of $F\backslash\{0\}$ and W be the submodule in F generated by G. Suppose that for all Λ_j, for all $v \in V(j, g_i, g_k)$ and for all $g_i,\ g_k \in G$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G. It suffices to show that for any $f \in W\backslash\{0\}$, there are $\lambda \in \Lambda$, $g \in G$ such that $\mathrm{lt}(f) = \mathrm{lt}(\lambda g)$.

Since W is generated by G, we have $f = \sum_{g \in G} h_g g$ for some $\{h_g\}_{g \in G} \subseteq D$.

Let $u = \max_\prec\{\mathrm{lt}(h_g g) | g \in G\}$. We may choose the family $\{h_g | g \in G\}$ such that u is minimal, i.e. if $f = \sum_{g \in G} h'_g g$ then $u \preceq \max_\prec\{\mathrm{lt}(h'_g g) | g \in G\}$. Note that $u \succeq \lambda g$ for all terms λ of h_g and all $g \in G$ by Proposition 3.6.

If $\mathrm{lt}(f) = u = \mathrm{lt}(h_g g)$ for some $g \in G$, then it is follows from (3.5) that there is a term λ of h_g such that $\mathrm{lt}(f) = \mathrm{lt}(\lambda g)$. Therefore the proof would be completed. Hence it remains to show that $\mathrm{lt}(f) \prec u$ cannot hold.

Suppose $\mathrm{lt}(f) \prec u$ and let $B = \{g|\mathrm{lt}(h_g g) = u \succ \mathrm{lt}(f)\}$. Then by (3.5) in the proof of Proposition 3.6, there is a unique term λ_g of h_g, $g \in B$, such that $u = \mathrm{lt}(\lambda_g g) \succ \mathrm{lt}(\eta_g g)$ for any terms $\eta_g \neq \lambda_g$ of h_g. Let c_g be the coefficient of h_g at λ_g. We have

$$f = \sum_{g\in B} h_g g + \sum_{g\notin B} h_g g = \sum_{g\in B} c_g\lambda_g g + \sum_{g\in B}(h_g - c_g\lambda_g)g + \sum_{g\notin B} h_g g, \tag{3.7}$$

where all terms appearing in the last two sums are $\prec u$.

From Lemma 3.4 (i), suppose v_g is the term of g such that $u = \mathrm{lt}(\lambda_g g) = \lambda_g v_g \succ \lambda_g v$ for any terms $v \neq v_g$ of g. Let d_g be the coefficient of g at v_g. Then by Lemma 3.3,

$$\begin{aligned}\sum_{g\in B} c_g\lambda_g g &= \sum_{g\in B} c_g\lambda_g d_g\left(\frac{g}{d_g}\right) = \sum_{g\in B} c_g(d'_g\lambda_g + \xi_g)\left(\frac{g}{d_g}\right)\\ &= \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) + \sum_{g\in B} c_g\xi_g\left(\frac{g}{d_g}\right)\end{aligned} \tag{3.8}$$

for some elements $d'_g \in K$ and $\xi_g \in D$ where all terms appearing in the last sum are $\prec u$.

Note that u appears only in

$$\begin{aligned}\sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) &= \sum_{g\in B} c_g d'_g\lambda_g v_g + \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g} - v_g\right)\\ &= \left(\sum_{g\in B} c_g d'_g\right)u + \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g} - v_g\right)\end{aligned}$$

and all terms appearing in the last sum are $\prec u$. Since $\mathrm{lt}(f) \prec u$ it follows that $\sum_{g\in B} c_g d'_g = 0$. Denote $\lambda_g(\frac{g}{d_g})$ by r_g, then by Lemma 3.15,

$$\sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) = \sum_{g\in B}(c_g d'_g)r_g = \sum_{i,k} b_{i,k}(r_{g_i} - r_{g_k}) \tag{3.9}$$

for some $g_i,\ g_k \in B$.

Since $r_{g_i} - r_{g_k} = \lambda_{g_i}(\frac{g_i}{d_{g_i}}) - \lambda_{g_k}(\frac{g_k}{d_{g_k}})$ and $\lambda_{g_i}v_{g_i} = \lambda_{g_k}v_{g_k} = u \in \Lambda_j E$ for some Λ_j, it follows from Lemma 3.14 that $v_{g_i} = \mathrm{lt}_j(g_i)$, $v_{g_k} = \mathrm{lt}_j(g_k)$, $d_{g_i} = \mathrm{lc}_j(g_i)$, $d_{g_k} = \mathrm{lc}_j(g_k)$, $\lambda_{g_i} = \frac{u}{\mathrm{lt}_j(g_i)}$, $\lambda_{g_k} = \frac{u}{\mathrm{lt}_j(g_k)}$ and then

$$r_{g_i} - r_{g_k} = \frac{u}{\mathrm{lt}_j(g_i)}\frac{g_i}{\mathrm{lc}_j(g_i)} - \frac{u}{\mathrm{lt}_k(g_k)}\frac{g_k}{\mathrm{lc}_j(g_k)}$$

with $\mathrm{lt}(r_{g_i} - r_{g_k}) \prec u$.

Note that for all Λ_j, for all $v \in$ and for all $g_i,\ g_k \in G$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G. Then by Lemma 3.16, we have

$$r_{g_i} - r_{g_k} = \sum_{g\in G} p_g g \tag{3.10}$$

with $\mathrm{lt}(p_g g) \prec u$.

Replace the first sum in the right side of (3.7) by (3.8), and replace the first sum in the r.h.s of (3.8) by (3.9), then substitute $r_{g_i} - r_{g_k}$ in the r.h.s of (3.9) by (3.10), we get another form of

$f = \sum_{g\in G} h'_g g$ and $u \succ \max_{\prec}\{\mathrm{lt}(h'_g g) | g \in G\}$, which is a contradiction to the minimality of u. This completes the proof of the theorem.

Example 3.18. If W is a submodule of F generated by a finite set G and every $g \in G$ consists of only one term, then G is a Gröbner basis of W. In fact in this case all S-polynomials $S(j, g_i, g_k, v)$ are 0. By Theorem 3.17 this implies that G is a Gröbner basis of W.

Theorem 3.19 (Buchberger's Algorithm)**.** *Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F\backslash\{0\}$ and W be the submodule in F generated by G. For each Λ_j and $f, g \in F\backslash\{0\}$ let $V(j, f, g)$ and $S(j, f, g, v)$ be as in Definition* 3.13. *Then by the following algorithm a Gröbner basis of W can be computed*:

Input: $G = \{g_1, \ldots, g_\mu\}$ *which is a set of generators of W*;

output: $G' = \{g'_1, \ldots, g'_\nu\}$ *which is a Gröbner basis of W*;

Begin

$G_0 := G$;

While *there exist $f, g \in G_i$ and $v \in V(j, f, g)$ such that $S(j, f, g, v)$ reduced to $r \neq 0$ by G_i*,

Do $G_{i+1} := G_i \cup \{r\}$;

End

Proof. By Theorem 3.17 we only have to show that there is an $i \in \mathbb{N}$ such that $G_{i+1} = G_i$. Suppose there is no such $i \in \mathbb{N}$. Then we have an infinite chain of sets $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_i \subsetneq \cdots$. Since there is a finite number of orthans Λ_j, $j = 1, \ldots, n$, we may assume that $\mathrm{lt}(r) \in \Lambda_j E$ in every G_{i+1} for a fixed j. Note that in every step of the algorithm we get r such that $\mathrm{lt}(r)$ does not equal any $\mathrm{lt}(\lambda g)$, $\lambda \in \Lambda$, $g \in G_i$. Also we have $\eta\mathrm{lt}(\lambda g) = \mathrm{lt}(\eta\lambda g) = \mathrm{lt}(\lambda' g) \in \Lambda_j E$ for any $\eta \in \Lambda_j$ and any $\mathrm{lt}(\lambda g) \in \Lambda_j E$ by Lemma 3.4 (ii). So if $\mathrm{lt}(r) \in \Lambda_j E$ then $K_j^{(i)} =_{K[\Lambda_j]} \langle \mathrm{lt}(\lambda g) \in \Lambda_j E | \lambda \in \Lambda,\ g \in G_i\rangle \subsetneq K_j^{(i+1)} =_{K[\Lambda_j]} \langle \mathrm{lt}(\lambda g) \in \Lambda_j E | \lambda \in \Lambda,\ g \in G_{i+1}\rangle$ as $K[\Lambda_j]$-submodule of $\bigoplus_{e\in E} K[\Lambda_j]e$. Therefore for all $i \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $K_j^{(i)} \subsetneq K_j^{(i+m)}$. Since $K[\Lambda_j]$ is noetherian, this is not possible.

Example 3.20. Let F and the generalized term order on Λ as in Example 3.14. Let $G = \{g_1, g_2, g_3\}$ and $g_1 = \alpha_2^4 + 1$, $g_2 = \alpha_1^2 - 1$, $g_3 = \alpha_1^2\alpha_2^4 + 1$. Then G is a Gröbner basis of W which is generated by G. To prove this, we show all S-polynomials of G reduced to 0 by G.

Following the method described in Example 3.14, we have

$$V(0, g_1, g_2) = \{\alpha_1^2\alpha_2^4\}, \quad V(1, g_1, g_2) = \{\alpha_1^{-1}\alpha_2^3\}, \quad V(2, g_1, g_2) = \{\alpha_1\alpha_2^{-1}\},$$

$$S(0, g_1, g_2, \alpha_1^2\alpha_2^4) = \alpha_1^2 g_1 - \alpha_2^4 g_2 = \alpha_1^2 + \alpha_2^4 = g_1 + g_2,$$

$$S(1, g_1, g_2, \alpha_1^{-1}\alpha_2^3) = \alpha_1^{-1}\alpha_2^{-1} g_1 + \alpha_1^{-1}\alpha_2^3 g_2 = \alpha_1^{-1}\alpha_2^{-1} + \alpha_1\alpha_2^3 = (\alpha_1^{-1}\alpha_2^{-1}) g_3,$$

$$S(2, g_1, g_2, \alpha_1\alpha_2^{-1}) = \alpha_1\alpha_2^{-1} g_1 - \alpha_1^{-1}\alpha_2^{-1} g_2 = \alpha_1^{-1}\alpha_2^{-1} + \alpha_1\alpha_2^3 = (\alpha_1^{-1}\alpha_2^{-1}) g_3$$

and

$$V(0, g_1, g_3) = \{\alpha_1^2\alpha_2^4\}, \quad V(1, g_1, g_3) = \{\alpha_1^{-1}\alpha_2^3\}, \quad V(2, g_1, g_3) = \{\alpha_2^{-1}\},$$

$$S(0, g_1, g_3, \alpha_1^2\alpha_2^4) = \alpha_1^2 g_1 - g_3 = \alpha_1^2 - 1 = g_2,$$

$$\begin{aligned} S(1, g_1, g_3, \alpha_1^{-1}\alpha_2^3) &= \alpha_1^{-1}\alpha_2^{-1} g_1 - \alpha_1^{-1}\alpha_2^3 g_3 = \alpha_1^{-1}\alpha_2^{-1} - \alpha_1\alpha_2^7 \\ &= (\alpha_1^{-1}\alpha_2^{-1}) g_3 - \alpha_1\alpha_2^3 g_1. \end{aligned}$$

Note that the r.h.s of this equation satisfies the condition in Theorem 3.7 (i), i.e. $\mathrm{lt}(h_i g_i) \preceq \mathrm{lt}(S)$, where $S = S(1, g_1, g_3, \alpha_1^{-1}\alpha_2^3)), i = 1, 3$.

$$S(2, g_1, g_3, \alpha_2^{-1}) = \alpha_2^{-1}g_1 - \alpha_2^{-1}g_3 = \alpha_2^3 - \alpha_1^2\alpha_2^3 = -\alpha_2^3 g_2.$$

Finally,

$$V(0, g_2, g_3) = \{\alpha_1^2\alpha_2^4\}, \quad V(1, g_2, g_3) = \{\alpha_1^{-1}\}, \quad V(2, g_2, g_3) = \{\alpha_1\alpha_2^{-1}\},$$

So

$$\begin{aligned} S(0, g_2, g_3, \alpha_1^2\alpha_2^4) &= \alpha_2^4 g_2 - g_3 = -\alpha_2^4 - 1 = -g_1, \\ S(1, g_2, g_3, \alpha_1^{-1}) &= \alpha_1^{-1}g_2 - \alpha_1^{-1}g_3 = \alpha_1\alpha_2^4 + \alpha_1 = \alpha_1 g_1, \\ S(2, g_2, g_3, \alpha_1\alpha_2^{-1}) &= \alpha_1^{-1}\alpha_2^{-1}g_2 - \alpha_1\alpha_2^{-1}g_3 = -\alpha_1^{-1}\alpha_2^{-1} - \alpha_1^3\alpha_2^3, \\ &= \alpha_1^{-1}\alpha_2^{-1}g_3 + \alpha_1\alpha_2^3 g_2. \end{aligned}$$

The r.h.s of this equation also satisfies the condition in Theorem 3.7 (i).

So, by Theorem 3.17, G is a Gröbner basis of W.

4 Applications to difference-differential dimension polynomials

In this section we describe a new approach to computing difference-differential dimension polynomials via the difference-differential Gröbner bases. There are some classical approaches described by many researchers (see Section 1). Our approach seems more general and more direct.

Let K be a Δ-σ-field, D the ring of Δ-σ-operators over K, M a finitely generated Δ-σ-module (i.e. a finitely generated difference-differential-module), F a finitely generated free Δ-σ-module. And we will keep the notation and conventions of the preceding sections.

For $\lambda \in \Lambda$ of the form (1.1), let $\mathrm{ord}\lambda = k_1 + \cdots + k_m + |l_1| + \cdots + |l_n|$. Also, for $w = \lambda e_i \in \Lambda E$ of a term of F, let ord $w = \mathrm{ord}\lambda$. If $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$, then ord $u = \max\{\mathrm{ord}\lambda | a_\lambda \neq 0\}$.

We may consider D as a filtered ring with the filtration $(D_\mu)_{\mu \in \mathbb{Z}}$ such that $D_\mu = \{u \in D | \mathrm{ord}\ u \leqslant \mu\}$ for any $\mu \in \mathbb{N}$ and $D_\mu = 0$ for $\mu < 0$. It is clear that $\bigcup\{D_\mu | \mu \in \mathbb{Z}\} = D$, $D_\mu \subseteq D_{\mu+1}$ for any $\mu \in \mathbb{Z}$ and $D_\nu D_\mu = D_{\mu+\nu}$ for any $\mu, \nu \in \mathbb{Z}$.

Definition 4.1. *Let K be a Δ-σ-field and M be a Δ-σ-module. A sequence $(M_\mu)_{\mu \in \mathbb{Z}}$ of K-vector subspaces of the module M is called a filtration of M if the following three conditions hold*:

(i) *$M_\mu \subseteq M_{\mu+1}$ for all $\mu \in \mathbb{Z}$ and $M_\mu = 0$ for all sufficiently small $\mu \in \mathbb{Z}$ (i.e. there is a $\mu_0 \in \mathbb{Z}$ such that $M_\mu = 0$ for all $\mu \leqslant \mu_0$)*;

(ii) $\bigcup\{M_\mu | \mu \in \mathbb{Z}\} = M$;

(iii) *$D_\nu M_\mu \subseteq M_{\mu+\nu}$ for any $\mu \in \mathbb{Z}, \nu \in \mathbb{N}$.*

If every K-vector space M_μ is of finite dimension and there exist numbers $\mu_0 \in \mathbb{Z}$ such that $D_\nu M_\mu = M_{\mu+\nu}$ for all $\mu \geqslant \mu_0$, $\nu \in \mathbb{N}$, then the filtration $(M_\mu)_{\mu \in \mathbb{Z}}$ is called an excellent filtration of M.

Example 4.2. Let M be a finitely generated Δ-σ-module (i.e. a left D-module) with generators $h_1, \ldots, h_q$. If $M_\mu = D_\mu h_1 + \cdots + D_\mu h_q$ for any $\mu \in \mathbb{Z}$, then $(M_\mu)_{\mu \in \mathbb{Z}}$ is an excellent filtration of M.

Definition 4.3. *Let K be a Δ-σ-field, M and N be two Δ-σ-modules over K. A homomorphism of K-modules $f: M \to N$ is called a Δ-σ-homomorphism (or difference-differential homomorphism), if $f(\beta x) = \beta f(x)$ for any $x \in M$, $\beta \in \Delta \cup \sigma^*$. Surjective (respectively, injective or bijective) Δ-σ-homomorphism is called a Δ-σ-epimorphism (respectively, Δ-σ-monomorphism or Δ-σ-isomorphism).*

Choose the canonical orthant decomposition on $\mathbb{Z}^n$ as in Example 2.2 and define the generalized term order "$\prec$" on ΛE of the terms of F as follows (see Example 2.10):

If $u = \delta_1^{k_1} \cdots \delta_m^{k_m} \alpha_1^{l_1} \cdots \alpha_n^{l_n} e_i$ and $v = \delta_1^{r_1} \cdots \delta_m^{r_m} \alpha_1^{s_1} \cdots \alpha_n^{s_n} e_j$, then

$$\begin{aligned} u \prec v \iff & (\text{ord } u, e_i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, l_1, \ldots, l_n) \\ & < (\text{ord } v, e_j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, s_1, \ldots, s_n) \text{ in lexicographic order.} \end{aligned}$$

Theorem 4.4. *Let K be a Δ-σ-field, D the ring of Δ-σ-operators over K and M be a finitely generated Δ-σ-module with generators $h_1, \ldots, h_q$. Let F be a free Δ-σ-module with a basis $e_1, \ldots, e_q$ and $\pi: F \to M$ the natural Δ-σ-epimorphism of F onto M (i.e. $\pi(e_i) = h_i$ for $i = 1, \ldots, q$).*

Let M_μ be the vector K-space as in Example 4.2. Suppose $G = \{g_1, \ldots, g_d\}$ is a Gröbner basis of $N = \ker \pi$ with respect to the generalized term order "$\prec$" defined above, U_μ is the set of all terms $w \in \Lambda E$ such that ord $w \leqslant \mu$ and $w \neq \text{lt}(\lambda g_i)$, $\lambda \in \Lambda$, $i = 1, \ldots, d$. Then $\pi(U_\mu)$ is a basis of the K-vector space M_μ.

Proof. First, we show that the set $\pi(U_\mu)$ generates the K-vector space $M_\mu = D_\mu h_1 + \cdots + D_\mu h_q$. Suppose $\lambda h_i \in M_\mu$ and $\lambda h_i \notin \pi(U_\mu)$ for some $i = 1, \ldots, q$, $\lambda \in \Lambda$, ord $\lambda \leqslant \mu$. Then $\lambda e_i \notin U_\mu$, whence $\lambda e_i = \text{lt}(\lambda' g_j)$ for some $\lambda' \in \Lambda$, $g_j \in G$. Therefore

$$\lambda' g_j = a_j \lambda e_i + \sum_\nu a_\nu \lambda_\nu e_\nu,$$

where $a_j \neq 0$ and $a_\nu \neq 0$ for finitely many a_ν. Obviously, $\lambda_\nu e_\nu \prec \lambda e_i$ and then ord $\lambda_\nu \leqslant \mu$. Since $G \subseteq N = \ker(\pi)$, we have $0 = \pi(g_j)$ and

$$0 = \lambda' \pi(g_j) = \pi(\lambda' g_j) = a_j \pi(\lambda e_i) + \sum_\nu a_\nu \pi(\lambda_\nu e_\nu) = a_j \lambda h_i + \sum_\nu a_\nu \lambda_\nu h_\nu.$$

So that λh_i is a finite linear combination with coefficients from K of some elements of the form $\lambda_\nu h_\nu$ $(1 \leqslant \nu \leqslant q)$ such that ord$\lambda_\nu \leqslant \mu$ and $\lambda_\nu e_\nu \prec \lambda e_i$. Thus, we can apply the induction to λe_j $(\lambda \in \Lambda,\ 1 \leqslant j \leqslant q)$ with respect to the order "$\prec$" and obtain that every element λh_i (ord$\lambda \leqslant \mu$, $1 \leqslant i \leqslant q$) can be written as a finite linear combination of elements of $\pi(U_\mu)$ with coefficients from K.

Now, let us prove that the set $\pi(U_\mu)$ is linearly independent over K. Suppose that $\sum_{i=1}^l a_i \pi(u_i) = 0$ for some $u_1, \ldots, u_l \in U_\mu$, $a_1, \ldots, a_l \in K$. Let $h = \sum_{i=1}^l a_i u_i$. Then $\pi(h) = 0$ and then $h \in N$. Since $\text{lt}(h) = u_i \in \{u_1, \ldots, u_l\}$, then $\text{lt}(h) \in U_\mu$ and then $\text{lt}(h) \neq \text{lt}(\lambda g_i)$, $\lambda \in \Lambda$, $i = 1, \ldots, d$, by the definition of U_μ. Since G is a Gröbner basis of N it follows from Proposition 3.11 (iii) that $h = 0$. Therefore $a_1 = \cdots = a_l = 0$. This completes the proof of the theorem.

From Theorem 4.4 the dimension of M_μ as a K-vector space can be computed by Gröbner bases of difference-differential modules.

Definition 4.5. *A polynomial $f(t_1,\dots,t_l)$ in l variables $t_1,\dots,t_l$ with rational coefficients is called numerical if $f(t_1,\dots,t_l)\in\mathbb{Z}$ for all sufficiently large $(r_1,\dots,r_l)\in\mathbb{Z}^l$, i.e. there exists a l-tuple $(s_1,\dots,s_l)\in\mathbb{Z}^l$ such that $f(r_1,\dots,r_l)\in\mathbb{Z}$ for all integers $r_1,\dots,r_l\in\mathbb{Z}$ with $r_i\geqslant s_i$ $(1\leqslant i\leqslant l)$.*

The following theorem proved by Levin[9] generalizes the Kondrateva's result on the numerical polynomials associated with subsets of $\mathbb{N}^m$ (cf. [17, 18]) to the numerical polynomials associated with subsets of $\mathbb{N}^m\times\mathbb{Z}^n$.

Theorem 4.6. *Let A be a subset of $\mathbb{N}^m\times\mathbb{Z}^n$. Choose the canonical orthant decomposition of $\mathbb{Z}^n$ (see Example 2.2). Let $\trianglelefteq$ be the partial order on $\mathbb{N}^m\times\mathbb{Z}^n$ such that $(k_1,\dots,k_m,l_1,\dots,l_n)\trianglelefteq(r_1,\dots,r_m,s_1,\dots,s_n)$ if and only if $(l_1,\dots,l_n)$ and $(s_1,\dots,s_n)$ belong to a same orthant and*

$$(r_1,\dots,r_m,|s_1|,\dots,|s_n|)\in\{(k_1,\dots,k_m,|l_1|,\dots,|l_n|)+\mathbb{N}^{m+n}\}.$$

Furthermore, let

$$W_A=\{w\in\mathbb{N}^m\times\mathbb{Z}^n|\textit{there is no element } a\in A \textit{ such that } a\trianglelefteq w\}$$

and

$$W_A[r,s]=\{(k_1,\dots,k_m,l_1,\dots,l_n)\in W_A|k_1+\cdots+k_m\leqslant r,|l_1|+\cdots+|l_n|\leqslant s\}.$$

Then there exists a numerical polynomial $\psi_A(t_1,t_2)$ in two variables t_1 and t_2 with the following properties:

(i) $\psi_A(r,s)=\mathrm{Card}\ W_A[r,s]$ *for all sufficiently large* $(r,s)\in\mathbb{N}^2$.

(ii) $\deg\ \psi_A\leqslant m+n$, $deg_{t_1}\psi_A\leqslant m$, *and* $\deg_{t_2}\psi_A\leqslant n$.

(iii) *If* $A=\emptyset$, *then* $\deg\ \psi_A=m+n$. *In this case,*

$$\psi_A(t_1,t_2)=\binom{t_1+m}{m}\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{n}{i}\binom{t_2+i}{i}.$$

(iv) $\psi_A(t_1,t_2)=0$ *if and only if* $(0,\dots,0)\in A$.

In [9] the author used Theorem 4.6 to prove the existence of difference-differential dimension polynomial $\psi(t_1,t_2)$ in two variables t_1,t_2 of the difference-differential module M by means of characteristic set with respect to a special reduction. But the approach of characteristic set is not valid for the one-variable case. However, our approach of Gröbner bases in difference-differential modules can deal with the difference-differential dimension polynomials in one variable effectively.

The analog of Theorem 4.6 for the existence of numerical polynomial $\phi_A(t)$ in one variable t associated with the subset A of $\mathbb{N}^m\times\mathbb{Z}^n$ can be obtained in the same way as that used in the proof of Theorem 4.6. (cf. [9]). We state it as follows.

Corollary 4.7. *Let A, $\trianglelefteq$ and W_A be the same as in the conditions of Theorem 4.6. Let*

$$W_A[\mu]=\{(k_1,\dots,k_m,l_1,\dots,l_n)\in W_A|k_1+\cdots+k_m+|l_1|+\cdots+|l_n|\leqslant\mu\}.$$

Then there exists a numerical polynomial $\phi_A(t)$ with the following properties:

(i) $\phi_A(\mu) = \text{Card } W_A[\mu]$ *for all sufficiently large* $\mu \in \mathbb{N}$.

(ii) $\deg\ \phi_A \leqslant m+n$, *and if* $A = \emptyset$ *then* $\deg\ \phi_A = m+n$.

(iii) $\phi_A(t) = 0$ *if and only if* $(0, \ldots, 0) \in A$.

Now we may use the approach of Gröbner bases of difference-differential modules to compute dimension polynomial in difference-differential modules. We give a definition after the following theorem.

Theorem 4.8. *Let K be a Δ-σ-field, D the ring of Δ-σ-operators over K and M be a finitely generated Δ-σ-module, and $(M_\mu)_{\mu\in\mathbb{Z}}$ an excellent filtration of M. Then there exists a numerical polynomial $\phi(t)$ such that* $\deg(\phi(t)) \leqslant m+n$ *and* $\phi(\mu) = \dim_R M_\mu$ *for all sufficiently large* $\mu \in \mathbb{N}$. *Furthermore, $\phi(t)$ can be written as* $\phi(t) = \frac{2^n a}{(m+n)!} t^{m+n} + o(t^{m+n})$, $a \in \mathbb{Z}$ *and $o(t^{m+n})$ denotes a polynomial from $\mathbb{Q}[t]$ whose degree is less than $m+n$, and the integers* $d = \deg\phi(t)$, a, *and* $\Delta^d\phi(t)$ *do not depend on the choice of a system of generators of the module M.* ($\Delta^d\phi(t)$ *denotes the d-th finite difference of* $\phi(t) : \Delta\phi(t) = \phi(t+1) - \phi(t)$, $\Delta^2\phi(t) = \Delta(\Delta\phi(t))$, *etc.*)

Proof. Since $(M_\mu)_{\mu\in\mathbb{Z}}$ is an excellent filtration of M it follows that every M_μ is a finitely generated K-vector space and $D_\nu M_\mu = M_{\mu+\nu}$ for $\mu \geqslant \mu_0$, $\nu \geqslant 0$. Let $h_1, \ldots, h_q$ be a basis of the K-vector space M_{μ_0}. Then the elements $h_1, \ldots, h_q$ generate M as a left D-module and $M_\mu = \sum_{i=1}^q D_{\mu-\mu_0} h_i$ for all $\mu \geqslant \mu_0$. Without loss of generality we can assume that $\mu_0 = 0$. (If $\phi(t)$ is a numerical polynomial with the desired properties that corresponds to the case $\mu_0 = 0$ then $\phi(t-\mu_0)$ is the one for arbitrary $\mu_0 \in \mathbb{Z}$.) Thus we may suppose that $M = \sum_{i=1}^q Dh_i$ and $M_\mu = \sum_{i=1}^q D_\mu h_i$ for all $\mu \in \mathbb{Z}$.

Let F be a free Δ-σ-module with a basis $e_1, \ldots, e_q$. Let $\pi : F \to M$, $N = \ker\pi$ and U_μ ($\mu \in \mathbb{N}$) be the same as in the conditions of Theorem 4.4. Furthermore, let "$\prec$" be the generalized term order on ΛE of the terms of F and $G = \{g_1, \ldots, g_d\}$ be the Gröbner basis of N as in Theorem 4.4. By Theorem 4.4, for any $\mu \in \mathbb{N}$, $\pi(U_\mu)$ is a basis of the K-vector space M_μ. Note that in the second part of the proof of Theorem 4.4 it was shown that the restriction of π on U_μ is bijective, we have $\dim_K M_\mu = \text{Card } \pi(U_\mu) = \text{Card}(U_\mu)$.

Note that $U_\mu = \{w \in \Lambda E | \text{ord } w \leqslant \mu;\ w \neq \text{lt}(\lambda g_i), \lambda \in \Lambda, g_i \in G\}$. Let $V_i^{(j)}$ be a finite set of generators of the $K[\Lambda_j]$-module ${}_{K[\Lambda_j]}\langle \text{lt}(\lambda g_i) \in \Lambda_j E | \lambda \in \Lambda\rangle$. Let $V = \bigcup_{i,j} V_i^{(j)}$. Then $U_\mu = \{w \in \Lambda E | \text{ord } w \leqslant \mu$; there is no $v \in V$ such that $v \trianglelefteq w\}$.

Let $V_{e_i} = \{v \in V | v = \lambda e_i, \lambda \in \Lambda\}$ and $U_\mu^{(i)} = \{w \in \Lambda e_i | \text{ord } w \leqslant \mu$; there is no $v \in V_{e_i}$ such that $v \trianglelefteq w\}$, $i = 1, \ldots, q$. Then $\text{Card}(U_\mu) = \sum_{i=1}^q \text{Card}(U_\mu^{(i)})$.

By Corollary 4.7, there exists a numerical polynomial $\phi_i(t)$ such that $\deg(\phi_i(t)) \leqslant m+n$ and $\phi_i(\mu) = \text{Card}(U_\mu^{(i)})$, $i = 1, \ldots, q$, for all sufficiently large $\mu \in \mathbb{N}$. Therefore $\phi(t) = \sum_{i=1}^q \phi_i(t)$ satisfies that $\deg(\phi(t)) \leqslant m+n$ and $\phi(\mu) = \text{Card}(U_\mu) = \dim_R M_\mu$ for all sufficiently large $\mu \in \mathbb{N}$.

The last conclusion of the theorem is well-known properties of the dimension polynomial $\phi(t)$ that satisfy that $\deg(\phi(t)) \leqslant m+n$ and $\phi(\mu) = \dim_R M_\mu$ for all sufficiently large $\mu \in \mathbb{N}$ (cf. [13]).

Definition 4.9. *The numerical polynomial $\phi(t)$ in Theorem* 4.8 *is called difference-differential dimension polynomial in one variable t associated with M.*

The difference-differential dimension polynomial is treated as characteristics of the system of defining equations on the generators of M and determines "strength" (Kondrateva[13] and Levin[9]) of the system of difference-differential equations.

In [15] the authors proved the existence of difference-differential dimension polynomials $\phi(t)$ associated with M by classical Gröbner basis methods of computation of Hilbert polynomials. Now we present an alternate direct and algorithmic approach of Gröbner bases on difference-differential modules to compute the difference-differential dimension polynomials. The following example shows that the computation of difference-differential dimension polynomials associated with M is rather simple by the method described in Theorem 4.8.

Example 4.10. Let K be a difference-differential field whose basic sets Δ and σ consist of a single derivation operator δ and a single automorphism α, respectively. Furthermore, let D be the ring of Δ-σ-operators over K and $M = Dh$ be a cyclic Δ-σ-module whose generator h satisfies the defining equation $(\delta^a\alpha^b + \delta^a\alpha^{-b} + \delta^{a+b})h = 0$. In other words, M is isomorphic to the factor module of a free Δ-σ-module F with a free generator e by its Δ-σ-submodule $N = D(\delta^a\alpha^b + \delta^a\alpha^{-b} + \delta^{a+b})e$. Let the generalized term order $\prec$ on ΛE be the same as in Theorem 4.8. Then $\{g = (\delta^a\alpha^b + \delta^a\alpha^{-b} + \delta^{a+b})e\}$ is a Gröbner basis of N (see Example 3.12). since $\mathrm{lt}(g) = (\delta^{a+b})e$ belongs to any ortant of ΛE, it follows from Lemma 3.4 (ii) that $\mathrm{lt}(\lambda g) = \lambda(\delta^{a+b})e$ for any $\lambda \in \Lambda$. Then by Theorem 4.8,

$$\dim_K M_t = \mathrm{Card}(U_t) = \mathrm{Card}\{u \in \Lambda | \mathrm{ord} u \leqslant t; u \neq \lambda\delta^{a+b}, \lambda \in \Lambda\}.$$

Therefore,

$$\begin{aligned}
\dim_K M_t &= \mathrm{Card}\{\delta^c\alpha^d | c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leqslant t, (c, |d|) \notin \{(a+b, 0) + \mathbb{N}^2\}\} \\
&= \mathrm{Card}\{\delta^c\alpha^d | c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leqslant t\} \\
&\quad - \mathrm{Card}\{\delta^c\alpha^d | c \in \mathbb{N}, d \in \mathbb{Z}, c + |d| \leqslant t - (a+b)\} \\
&= [(t+2)(t+1) - (t+1)] - [(t-a-b+2)(t-a-b+1) - (t-a-b+1)] \\
&= 2(a+b)t + (a+b)(2-a-b).
\end{aligned}$$

The result of above example coincides with that shown in [9, 13]. But our approach is based on computing the Gröbner bases on difference-differential modules directly. The following example shows that when we choose another generalized term order to compute the Gröbner bases we can also get the same difference-differential dimension polynomial.

Example 4.11. Let M be the Δ-σ-module same as in Example 4.10. But the generalized term order $\prec$ on ΛE is defined as follows:

$$\delta^k\alpha^l e \prec \delta^r\alpha^s e \Longleftrightarrow (k + |l|, |l|, k, l) < (r + |s|, |s|, r, s) \ \text{ in lexicographic order.}$$

Note that Theorems 4.4 and 4.8 still valid for "$\prec$". Denote $\{\delta^k\alpha^l | l \geqslant 0\}$ by Λ_1 and $\{\delta^k\alpha^l | l \leqslant 0\}$ by Λ_2. Since $\mathrm{lt}(g) = \delta^a\alpha^b e \in \Lambda_1$ and $\mathrm{lt}(\alpha^{-1}g) = \delta^a\alpha^{-(b+1)}e \in \Lambda_2$ it follows that

$$\{\mathrm{lt}(\lambda g) \in \Lambda_1 | \lambda \in \Lambda\} = \Lambda_1\delta^a\alpha^b e, \qquad \{\mathrm{lt}(\eta g) \in \Lambda_2 | \eta \in \Lambda\} = \Lambda_2\delta^a\alpha^{-(b+1)}e.$$

Therefore

$$\dim_K M_t = \mathrm{Card}\{\delta^c\alpha^d | c, d \in \mathbb{N}, c + d \leqslant t, (c, d) \notin \{(a, b) + \mathbb{N}^2\}\}$$

$$
\begin{aligned}
&+\mathrm{Card}\{\delta^c\alpha^d | c\in\mathbb{N}, d\in\mathbb{Z}, d<0, c+|d|\leqslant t, (c,-d)\notin\{(a,b+1)+\mathbb{N}^2\}\}\\
=&\left[\frac{1}{2}(t+1)(t+2)-\frac{1}{2}(t-a-b+1)(t-a-b+2)\right]\\
&+\left[\frac{1}{2}t(t+1)-\frac{1}{2}(t-a-b)(t-a-b+1)\right]\\
=&\,2(a+b)t+(a+b)(2-a-b).
\end{aligned}
$$

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